Heat or Mass Transfer in a Fluid in Laminar Flow in a Circular or Flat Conduit

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Accurate solutions to the Graetz equation and to the similar equation for flow between two parallel plates are presented including the first ten or eleven eigenvalues and important derivatives. The first six eigenfunctions are also presented at intervals of 0.05 from y = 0 to y = 1.

The similar problems of steady state conduction of heat and the steady state diffusion in a fluid flowing in a circular or flat duct have been studied by many investigators. The Fourier-Poisson equation for steady state heat conduction (*11, 28*) may be written

$$V \cdot \nabla t = \alpha \nabla^2 t \quad (1)$$

or

$$V_x \frac{\partial t}{\partial x} + V_y \frac{\partial t}{\partial y} + V_z \frac{\partial t}{\partial z} = \alpha \left[\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \right] \quad (1a)$$

For one directional flow in a round tube with axial symmetry this equation becomes

$$\frac{V}{\alpha}\frac{\partial t}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial t}{\partial r} \right) + \frac{\partial^2 t}{\partial x^2} \quad (2)$$

In laminar flow the velocity distribution is parabolic, and $V = V\max [1-(r/R)^2]$; $(\partial^2 t/\partial x^2)$ is usually so small in comparison with the other terms that it can be neglected without much error. Equation (2) can therefore be written in dimensionless form as

$$(1-y^2)\frac{\partial T}{\partial z} = \frac{1}{y}\frac{\partial}{\partial y}\left(y \frac{\partial T}{\partial y} \right) \quad (3)$$

where $z = 2\alpha x/D^2 V_{AV} = 2x/DN_{Re}N_{Pr}$
$y = r/R$
$T = (t-t_s)/(t_o-t_s)$

For one directional flow in a flat duct (between two parallel plates) the corresponding equation is

$$\frac{3}{8}(1-y^2)\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial y^2} \quad (4)$$

where $z = \alpha x/4R^2 V_{av} = x/R\, N_{Re} N_{Pr}$
The corresponding equations for mass transfer by diffusion are usually written with concentration or partial pressure in place of temperature, and diffusivity in place of thermal diffusivity.

The first published solution to Equation (3) was presented by Graetz (*13, 14*). The assumptions and boundary conditions made by Graetz are constant thermal diffusivity, constant tube wall temperature, temperature symmetrical about the axis, uniform temperature at tube inlet, fully developed parabolic velocity profile at tube inlet, and negligible conduction in direction of flow.

These assumptions correspond to the following boundary conditions:

1. $T = 1 \quad 0 \leqq y \leqq 1 \quad -\infty \leqq z < 0$
2. $T = 0 \quad y = 1 \quad 0 < z \leqq +\infty$
3. $\partial T/\partial y = 0 \quad y = 0 \quad -\infty \leqq z \leqq +\infty$
4. $T = 0 \quad 0 \leqq y \leqq 1 \quad z = +\infty$

This solution to Equation (3) may be written as

$$T = \sum_{n=1}^{n=\infty} C_n Y_n \exp(-\lambda_n^2 z) \quad (5)$$

where λ_n is the n^{th} eigenvalue which is necessary for the solution to the differential equation

$$\frac{\partial^2 Y}{\partial y^2} + \frac{1}{y}\frac{\partial Y}{\partial y} + \lambda^2(1-y^2)Y = 0 \quad (6)$$

corresponding to the boundary conditions $Y = 0$ at $y = 1$, and $\partial Y/\partial y = 0$ at $y=0$. The coefficients C_n, which make Equation (5) satisfy the boundary condition $T = 1$ at $z = 0$ and $0 \leqq y < 1$, are obtained from

$$T = 1 = \sum_{n=1}^{n=\infty} C_n Y_n \quad (7)$$

by multiplying both sides of Equation (7) by $Y_m y(1-y^2)dy$ and integrating from $y = 0$ to $y = 1$. This procedure results in

$$C_n = \frac{\int_0^1 Y_n y(1-y^2)dy}{\int_0^1 Y_n^2 y(1-y^2)dy} = \frac{-2}{\lambda_n \left(\frac{\partial Y_n}{\partial \lambda_n} \right)_{y=1}} \quad (8)$$

which comes from

$$\int_0^1 Y_n Y_m y(1-y^2)dy = 0 \quad \text{if } n \neq m$$

$$\int_0^1 Y_n^2 y(1-y^2)dy = \frac{1}{2\lambda_n}\left(\frac{\partial Y_n}{\partial y} \cdot \frac{\partial Y_n}{\partial \lambda_n} \right)_{y=1} \quad \text{if } n = m$$

$$\int_0^1 Y_n y(1-y^2)\,dy = -\frac{1}{\lambda_n^2}\left(\frac{\partial Y}{\partial y} \right)_{y=1}$$

The function $Y_n(y)$ may be represented by the infinite series

$$Y_n = \sum_{i=0}^{i=\infty} a_{ni} y^i \quad (9)$$

where $a_{ni} = 0$ if $i < 0$
$a_{ni} = 1$ if $i = 0$
$a_{ni} = -\lambda_n^2 (a_{i-2} - a_{i-4})/i^2$

or the equivalent series

$$Y_n = \sum_{i=0}^{i=\infty} \sum_{j=i}^{j=2i} A_{ij} \lambda_n^{2i} y^{2j} \quad (10)$$

TABLE 1. EIGENVALUES AND DERIVATIVES—ROUND CASE

n	λ_n	$(\partial Y_n/\partial \lambda)_{y=1}$	$(\partial Y_n/\partial y)_{y=1}$
1	2.70436 44199	-0.50089 91914	-1.01430 04587
2	6.67903 14493	0.37146 22734	1.34924 16221
3	10.67337 95381	-0.31826 44696	-1.57231 93392
4	14.67107 84627	0.28648 21001	1.74600 43350
5	18.66987 18645	-0.26449 06034	-1.89085 71240
6	22.66914 33588	0.24799 44920	2.01646 66530
7	26.66866 19960	-0.23496 76067	-2.12816 47501
8	30.66832 33409	0.22430 62663	2.22925 54182
9	34.66807 38224	-0.21534 85062	-2.32194 33391
10	38.66788 33469	0.20766 87724	2.40778 11647
11	42.66773 38055	-0.20097 87384	-2.48790 82547

TABLE 2. EIGENFUNCTIONS—ROUND CASE

y	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	y
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	0.00
0.05	0.99543708	0.97232998	0.93009952	0.87000981	0.79384934	0.70387285	0.05
0.10	0.98184469	0.89180935	0.73545009	0.53108099	0.30228888	0.07488082	0.10
0.15	0.95950842	0.76560457	0.45728832	0.11310661	−0.18271103	−0.36382498	0.15
0.20	0.92889268	0.60469973	0.15247311	−0.23303152	−0.40260123	−0.32121955	0.20
0.25	0.89062392	0.42260986	−0.12055081	−0.39912334	−0.29328114	0.03117856	0.25
0.30	0.84546827	0.23385711	−0.31521322	−0.35914102	0.00054293	0.28981991	0.30
0.35	0.79430462	0.05242653	−0.40611068	−0.16904247	0.24974768	0.22529155	0.35
0.40	0.73809441	−0.10959274	−0.39208452	0.06793183	0.29907438	−0.04765772	0.40
0.45	0.67784945	−0.24301831	−0.29305410	0.24985073	0.14844781	−0.24623845	0.45
0.50	0.61459912	−0.34214076	−0.14234190	0.31507157	−0.07973259	−0.20531795	0.50
0.55	0.54935825	−0.40482321	0.02279222	0.25703015	−0.24057455	0.00683579	0.55
0.60	0.48309693	−0.43218156	0.16968455	0.11416883	−0.25522963	0.19749710	0.60
0.65	0.41671327	−0.42794545	0.27593866	−0.05472615	−0.13791706	0.22889512	0.65
0.70	0.35100978	−0.39762949	0.33148788	−0.19604286	0.03610027	0.10372100	0.70
0.75	0.28667401	−0.34764861	0.33742641	−0.27757775	0.18482648	−0.07719162	0.75
0.80	0.22426362	−0.28449432	0.30272281	−0.29224076	0.25918362	−0.20893147	0.80
0.85	0.16419583	−0.21405616	0.24015828	−0.25200258	0.25273850	−0.24408867	0.85
0.90	0.10674088	−0.14113350	0.16262482	−0.17762079	0.18817264	−0.19521652	0.90
0.95	0.05201900	−0.06914371	0.08046102	−0.08916322	0.09629593	−0.10234372	0.95
1.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	1.00

where

$$A_{ij} = 0 \text{ if } j<i,\ j>2i, \text{ or } i<o$$

$$A_{ij} = 1 \quad \text{if } i = j = o$$

and in general

$$A_{ij} = \frac{A_{i-1,\,j-2} - A_{i-1,\,j-1}}{(2j)^2}$$

These coefficients A_{ij} have been calculated for $0 \leqq i \leqq 7$ and $0 \leqq j \leqq 14$.*

C_n and λ_n of Equation (5) have previously been calculated for only the first five values of n and to only a few significant figures (*1, 5, 7, 13, 14, 17, 20, 21, 22, 23, 27, 30, 34*). Y_n has been given only for n = 1, 2, 3, 4 (*17, 21, 27, 34*). Expressions have been derived for approximations to all the eigenvalues and related constants, which become increasingly accurate as $n \to \infty$ (*20, 31*). The need for accurate values for the eigenvalues and functions has recently been shown (*24, 33*).

SOLUTION TO GRAETZ EQUATION FOR ROUND DUCT

The eigenvalues of the Graetz equation can be calculated from Equation (9) by a trial-and-error procedure. A trial value of λ is used to obtain the coefficients, and Y is obtained for the case $y = 1$. In addition to the calculation of Y the two partial derivatives $(\partial Y/\partial \lambda)_{y=1}$ and $(\partial Y/\partial y)_{y=1}$ are also calculated. The calculation of these quantities to any degree of accuracy with a desk calculator is extremely laborious, even for the first few eigenvalues, and practically impossible for values beyond about $n = 5$. This is due to the fact that the first few terms of the infinite series become progressively larger (until about the term $i = \sqrt{2}\lambda$) before becoming smaller. The largest term in this series is equal to about 10^{2n}, and the number of terms which have to be calculated is about $n^2 + m$ (where m is about equal to the number of significant figures desired in the answers).

To obtain accurate eigenvalues and related derivatives the computation was done on an IBM 650 electronic computer. The computation program was designed to handle terms which contained up to fifty significant figures (twenty to the left of the decimal point and thirty to the right). This program was used to compute λ_n, λ_n^2, $(\partial Y_n/\partial \lambda_n)_{y=1}$, $(\partial Y_n/\partial y)_{y=1}$, and Y_n vs. y with an accuracy of about thirty significant figures for values of n up to 11.

The method of computation included calculation of Y_n and its derivatives by means of the following series:

$$Y_n]_{y=1} = a_o + a_2 + a_4 + a_6 + \ldots a_i + \ldots \tag{11}$$

where

$$a_i = -\lambda_n^2 (a_{i-2} - a_{i-4})/i^2$$

$$\partial Y/\partial \lambda]_{y=1} = \frac{\partial a_2}{\partial \lambda} + \frac{\partial a_4}{\partial \lambda} + \frac{\partial a_i}{\partial \lambda} + \ldots \tag{12}$$

where

$$\frac{\partial a_i}{\partial \lambda} = \frac{-2\lambda}{i^2}(a_{i-2} - a_{i-4}) - \frac{\lambda^2}{i^2}\left[\frac{\partial a_{i-2}}{\partial \lambda} - \frac{\partial a_{i-4}}{\partial \lambda}\right]$$

$$\frac{\partial a_o}{\partial \lambda} = 0$$

$$(\partial Y/\partial y)_{y=1} = 2a_2 + 4a_4 + \ldots ia_i + \ldots \tag{13}$$

The series was continued until $a_i < 10^{-30}$ and the sums taken. From each trial calculation were obtained Y, $\partial Y/\partial \lambda$, and $\partial Y/\partial y$ (all for $y = 1$). From these values the next trial $\lambda_{n,j+1}$ was obtained from the previous trial value $\lambda_{n,j}$, by

$$\lambda_{n,j+1} = \lambda_{n,j} - \frac{Y}{\partial Y/\partial \lambda} \tag{14}$$

By this procedure it was possible to obtain eigenvalues accurate to about thirty significant figures with as few as four trials. The results of these calculations rounded to ten decimal places are given in Table 1.

After each eigenvalue was determined with sufficient accuracy, Y_n was calculated for different values of y.

* Tabular material has been deposited as document No. 6126 with the American Documentation Institute, Photoduplication Service, Library of Congress, Washington 25, D. C., and may be obtained for $1.25 for photoprints or $1.25 for 35-mm. microfilm.

TABLE 3. COMPARISON OF EIGENVALUES—ROUND CASE

		λ_1	λ_2	λ_3	λ_4	λ_5
Graetz	(13,14)	2.7043	6.50			
Nusselt	(27)	2.705	6.66	10.3		
Drew	(7)	2.70436	6.6791			
Lee	(21)	2.704	6.679	10.673	14.671	
Yamagata	(34)	2.704365	6.67903	10.67340	14.6712	18.67
Brinkman	(5)	(2.704)*	(6.679)	(10.673)	(14.63)	
Lauwerier	(20)	(2.70434)*	(6.680)	(10.6734)	(14.67108)	
Schenk	(30)	2.70437	6.6790	10.6733	14.6711	18.670
Abramowitz	(1)	2.7043644	6.679032	10.67338	14.67108	18.66987
Sellars	(31)	2 2/3	6 2/3	10 2/3	14 2/3	18 2/3
Sellars	(31)	2.71	6.69	10.62	14.58	
Lipkis	(23)	2.7043644	6.679032	10.67338	14.67108	18.66987
Brown		2.7043644199	6.6790314493	10.6733795381	14.6710784627	18.6698718645

*Values in parentheses were calculated from data reported in reference.

These calculations were made from the terms of Equation (11) already calculated by the use of Equation (9)

$$Y_n = a_o + a_2 y^2 + a_4 y^4 + \ldots \quad (9)$$

for values of y between 0 and 1 at increments of 0.05. Table 2 gives these results for the first six eigenfunctions rounded to eight decimal places.

Equation (5) gives T as a function of y and z. The average or cup-mixing temperature is

$$T_{AV} = \frac{{}_0\int^1 T\, V\, 2\pi\, y\, dy}{\pi\, V_{av}}$$

Since

$$V = 2\, V_{AV}(1-y^2)$$

$$T_{AV} = {}_0\int^1 4T\,(1-y^2)\, y dy$$

and

$$T_{AV} = \sum_{n=1}^{n=\infty} \left[\frac{-8}{\lambda_n \left(\frac{\partial Y_n}{\partial \lambda_n}\right)_{y=1}} \cdot [\exp(-\lambda_n^2 z)]\, {}_0\int^1 Y_n y (1-y^2) dy \right]$$

$$T_{AV} = \sum_{n=1}^{n=\infty} \frac{8\left(\frac{\partial Y_n}{\partial y}\right)_{y=1}}{\lambda_n^3 \left(\frac{\partial Y_n}{\partial \lambda_n}\right)_{y=1}} \exp(-\lambda_n^2 z) \quad (15)$$

COMPARISON OF RESULTS

Equations (5), (8), and (15) have previously been derived by several writers (*4, 5, 7, 14, 15, 17, 27, 34*), and their applications to heat transfer (*8, 9, 10, 17, 18, 19, 24, 26, 31, 33, 34*) and mass transfer (*2, 3, 6, 12, 32*) have been discussed. A comparison of the eigenvalues with those previously calculated is given in Table 3. A comparison of the eigenfunctions $(\partial Y/\partial \lambda)$ and $(\partial Y/\partial y)$ with those previously calculated has been made.*

* See footnote on page 180.

The function $Y_n(y)$ has been previously calculated (*17, 21, 27, 34*) to only three or four significant figures and for only the first four values of n.

Although values of Y_n in Table 2 are given for only the first six values of n, at increments of 0.05, and reported to only eight significant figures, the program is capable of calculating Y_n for any n up to 11, any y between 0 and 1, and with an accuracy of nearly thirty significant figures. Results are now available for $n = 1$ to $n = 5$, $y = 0$ to $y = 1$ with increments of 0.02 with Y to thirty places.

SOLUTION TO EQUATION FOR FLAT DUCT

Equation (4) and its solution for the case with the same boundary conditions as for Equation (5) has been derived (*29, 31*). This solution is

$$T = \sum_{f=1}^{f=\infty} C_f \cdot Y_f \cdot \exp\left(-\frac{8}{3}\lambda_f^2 z\right) \quad (16)$$

where λ_f is the f^{th} eigenvalue, which is necessary for the solution to the differential equation

$$\frac{\partial^2 Y}{\partial y^2} + \lambda^2 (1-y^2) Y = 0 \quad (17)$$

corresponding to the boundary conditions $Y = 0$ at $y = 1$, $\partial Y/\partial y = 0$ at $y = 0$. The C_f which make Equation (16) satisfy the boundary condition $T = 1$ at $z = 0$ and $-1 \leq y \leq 1$ are obtained from

$$T = 1 = \sum_{f=1}^{f=\infty} C_f Y_f \quad (18)$$

Multiplying both sides by $Y_m(1-y^2)dy$ and integrating from $y = 0$ to $y = 1$ one obtains

TABLE 4. EIGENVALUES AND DERIVATIVES—FLAT CASE

f	λ_f	$(\partial Y_f/\partial\lambda)_{y=1}$	$(\partial Y_f/\partial y)_{y=1}$
1	1.68159 53222	-0.99043 69608	-1.42915 55060
2	5.66985 73459	1.17910 73461	3.80707 01070
3	9.66824 24625	-1.28624 87056	-5.92023 79188
4	13.66766 14426	1.36201 96175	7.89253 51208
5	17.66737 35653	-1.42132 56612	-9.77094 42849
6	21.66720 53243	1.47040 11597	11.57980 87072
7	25.66709 64863	-1.51246 03349	-13.33387 89738
8	29.66702 10447	1.54938 60066	15.04298 83445
9	33.66696 60687	-1.58238 01630	-16.71412 93950
10	37.66692 44563	1.61225 92197	18.35251 24063

TABLE 5. EIGENFUNCTIONS—FLAT CASE

y	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	y
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	0.00
0.05	0.99646885	0.96010074	0.88546018	0.77552799	0.63468541	0.46854394	0.05
0.10	0.98591788	0.84377192	0.56853419	0.20356398	−0.19367528	−0.56064196	0.10
0.15	0.96847352	0.66076030	0.12249535	−0.45940625	−0.88289598	−1.00084872	0.15
0.20	0.94434300	0.42615789	−0.35125799	−0.92127048	−0.94139939	−0.39858805	0.20
0.25	0.91380933	0.15883318	−0.74804895	−0.98588465	−0.33711948	0.61543949	0.25
0.30	0.87722437	−0.12047066	−0.98430775	−0.63425823	0.50149201	1.01611220	0.30
0.35	0.83500029	−0.39113244	−1.01518005	−0.02144606	1.00745915	0.42451165	0.35
0.40	0.78759965	−0.63450596	−0.84140636	0.60158620	0.86317657	−0.57373954	0.40
0.45	0.73552430	−0.83542525	−0.50482523	0.99796998	0.18131026	−1.05774223	0.45
0.50	0.67930340	−0.98321739	−0.07498014	1.03501041	−0.61211784	−0.62716953	0.50
0.55	0.61948097	−1.07215385	0.36866934	0.71960900	−1.06882651	0.32311352	0.55
0.60	0.55660303	−1.10134778	0.75396988	0.17433415	−0.97130705	1.03602948	0.60
0.65	0.49120483	−1.07417239	1.02875272	−0.42445889	−0.40612604	1.02291537	0.65
0.70	0.42379830	−0.99732402	1.16687578	−0.91420090	0.34092183	0.35052804	0.70
0.75	0.35485987	−0.87967811	1.16760186	−1.19276885	0.96322186	−0.53240219	0.75
0.80	0.28481910	−0.73108692	1.04990111	−1.23289457	1.27030674	−1.16513731	0.80
0.85	0.21404783	−0.56124735	0.84407970	−1.06843948	1.23140209	−1.32944153	0.85
0.90	0.14285022	−0.37873136	0.58311581	−0.76562278	0.92849439	−1.07197987	0.90
0.95	0.07145363	−0.19022795	0.29544440	−0.39311565	0.48542365	−0.57342785	0.95
1.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	1.00

$$C_f = \frac{\int_0^1 Y_f\,(1-y^2)\,dy}{\int_0^1 Y_f^2\,(1-y^2)\,dy} = \frac{-2}{\lambda_f\left(\dfrac{\partial Y_f}{\partial \lambda_f}\right)_{y=1}} \qquad (19)$$

The eigenvalues and related functions can be easily calculated from the infinite series

$$Y_f = \sum_{i=0}^{i=\infty} b_{fi}\, y^i \qquad (20)$$

in exactly the same manner as was done for the round duct with a single change. The recursion formulas for b_f and $\partial b/\partial\lambda$ are slightly different from those for a_n in that i^2 is replaced by $i(i-1)$. These formulas are

$$b_{f,i} = -\lambda_f^2(b_{f,\,i-2} - b_{f,\,i-4})/i(i-1)$$

$$\frac{\partial b_i}{\partial \lambda} = \frac{-2\lambda}{i(i-1)}\,(b_{i-2} - b_{i-4}) -$$

TABLE 6. COMPARISON OF EIGENVALUES AND DERIVATIVES—FLAT CASE

f		1	2	3	4	5	6
λ_f							
Prins	(29)	1.6816	5.6699	9.6678			
Schenk	(30)	1.6816	5.6699	9.6678	13.6677	17.667	
Sellars	(31)	1 2/3	5 2/3	9 2/3	13 2/3	17 2/3	21 2/3
Harris	(16)	1.68159532	5.66985734	9.66824246	13.66766144	17.66737357	21.66720532
Brown		1.6815953222	5.6698573459	9.6682424625	13.6676614426	17.6673735653	21.6672053243
$(\partial Y/\partial\lambda)$							
Prins	(29)	-0.990	1.21	-1.35			
Sellars	(31)	(-0.95887185)*	(1.17581877)	(-1.28528242)	(1.36164181)	(-1.42116624)	(1.47034008)
Harris	(16)	-0.99043694	1.17910684	-1.28624824	1.36201795	-1.42132438	1.47039947
Brown		-0.9904369608	1.1791073461	-1.2862487056	1:3620196175	-1.4213256612	1.4704011597
$(\partial Y/\partial y)$							
Prins	(29)	-1.434	3.86	-5.9			
Sellars	(31)	(-1.36514243)*	(3.78509744)	(-5.90701347)	(7.88296340)	(-9.76336489)	(11.57348340)
Harris	(16)	-1.429155504	3.807070140	-5.920237920	7.892535140	-9.770944400	11.57980880
Brown		-1.4291555060	3.8070701070	-5.9202379188	7.8925351208	-9.7709442848	11.5798087072

*Values in parentheses were calculated from equations given by Sellars (31), who also reported incorrect values for C_f and $-C_f\,(\partial Y/\partial y)$, which should be multiplied by 2.5. The corrected equations are:

$$C_f = (-1)^f\, 2.27114\, \lambda_f^{-7/6} \qquad \text{and} \qquad -C_f\,(\partial Y/\partial y) = 2.02557\, \lambda_f^{-1/3}$$

$$\frac{\lambda^2}{i(i-1)}\left(\frac{\partial b_{i-3}}{\partial\lambda}-\frac{\partial b_{i-4}}{\partial\lambda}\right)$$

The resulting calculated values rounded to ten decimal places for the eigenvalues λ_f and the derivatives $(\partial Y_f/\partial\lambda_f)_{y=1}$ and $(\partial Y_f/\partial y)_{y=1}$ are given in Table 4 for $f = 1$ to $f = 10$.

After the eigenvalues were obtained, the first six eigenfunctions Y_f were calculated for the interval $y = 0$ to $y = 1$ from Equation (20)

$$Y_f = b_o + b_2y^2 + b_4y^4 + \ \ldots . \quad (20)$$

with increments of 0.05. These results were calculated to thirty decimal places and values rounded to eight places are given in Table 5.

Equation (16) gives T as a function of y and z. The average or cup-mixing temperature is

$$T_{AV} = \frac{{}_0\!\int^1 T\,V\,dy}{V_{AV}}$$

Since

$$V = \frac{3}{2}V_{AV}(1-y^2)$$

$$T_{AV} = {}_0\!\int^1 \frac{3}{2}T\,(1-y^2)dy$$

$$T_{av} = \frac{3}{2}\sum_{f=1}^{f=\infty}\left[\frac{-2}{\lambda_f\left(\frac{\partial Y_f}{\partial\lambda_f}\right)_{y=1}}\exp\left(-\frac{8}{3}\lambda_f^2 z\right)\,{}_0\!\int^1 Y_f(1-y^2)dy\right]$$

$$T_{AV} = \sum_{f=1}^{f=\infty}\frac{3\left(\frac{\partial Y_f}{\partial y}\right)_{y=1}}{\lambda_f^3\left(\frac{\partial Y_f}{\partial\lambda_f}\right)_{y=1}}\exp\left(-\frac{8}{3}\lambda_f^2 z\right) \quad (21)$$

These eigenvalues, λ_f, and the derivatives $(\partial Y_f/\partial\lambda_f)_{y=1}$ and $(\partial Y_f/\partial y)_{y=1}$ have been previously estimated by an asymptotic solution (*31*) and calculated by a series expansion (*16, 29, 30*). A comparison with these values is given in Table 6. The values for the eigenfunctions Y_f in Table 5 are in substantial agreement with values previously calculated (*16, 29*) for $f = 1$ to $f = 6$ with increments in y of 0.01 and accurate to eight significant figures.

CONCLUSION

The eigenvalues and derivatives presented in Table 1 for the Graetz problem and in Table 4 for the similar problem in a flat duct together with the asymptotic values (*31*) for $n > 11$ and $f > 10$ should be accurate enough for all purposes which require the stated boundary conditions. For other boundary conditions and different velocity distributions the problem solutions can be obtained by modification of the computation program.

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NOTATION

A	=	coefficient in Graetz equation, Equation (10)
a	=	coefficient in Graetz equation, Equation (9)
b	=	coefficient, Equation (20)
C	=	coefficient, Equation (8)
C_p	=	heat capacity
D	=	diameter of conduit, (=2R)
k	=	thermal conductivity
R	=	radius of conduit or half width of flat duct
r	=	distance from center of duct
t	=	temperature
T	=	temperature, dimensionless $[=(t-t_s)/t_o-t_s)]$
V	=	velocity
Y	=	functions, Equation (6)
x, y, z	=	Cartesian coordinates
y	=	relative coordinate, (=r/R)
z	=	relative longitudinal coordinate, $(=x/R\,N_{Re}\,N_{Pr})$
α	=	thermal diffusivity, $(=k/C_p\rho)$
λ	=	Eigenvalue, Equations (6), (17)
μ	=	viscosity
ρ	=	density

Dimensionless Groups

N_{Pr}	=	Prandtl number, $(= C_p\,\mu/k)$
N_{Re}	=	Reynolds number, $(= DV_{AV}\,\rho/\mu$ for round duct; $= 4R\,V_{AV}\,\rho/\mu$ for flat duct)

Subscripts

i, j	=	integers, Equation (9)
f, n, m	=	integers
s	=	conduit wall
o	=	conduit entrance
x, y, z	=	Cartesian coordinate components
av	=	average
max	=	maximum
V_{AV}	=	average velocity, $(=V_{max}/2$ for round duct; $= 2V_{max}/3$ for flat duct)

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